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A functional limit theorem for trimmed sums

Yuji Kasahara
Institute of Mathematics, University of Tsukuba, Ibaraki, Japan

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This paper proves a functional limit theorem for Stigler's result on the heavily trimmed sums of i.i.d. random variables. The limiting process will be expressed as a functional of a Kiefer process and we shall also see that it is a Brownian motion if and only if asymptotic normality holds.

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1. Introduction

This paper is essentially a continuation of Kasahara and Maejima [2], where we discussed a functional limit theorem for heavily trimmed sums of the Stigler type. What is new in the present paper is that we shall be interested in the time parameter, while the previous paper [2] is concerned with the space parameter, instead.

Let $\{X_j\}_{j=1}^{\infty}$ be a sequence of i.i.d. random variables with a common distribution function $F(x)$ and for each $n\geqslant 1$ let $X_1^{(n)}\leqslant X_2^{(n)}\leqslant \cdots \leqslant X_n^{(n)}$ denote the order statistics of $\{X_1, X_2, \ldots, X_n\}$. Let a, b $(0\leqslant a<b\leqslant 1)$ be fixed numbers and consider the so called heavily trimmed sum

$$T_n = T_n(a, b) := \sum_{j=[na]+1}^{[nb]} X_j^{(n)}, \quad n\geqslant 0, \tag{1.1}$$

where $[x]$ denotes the greatest integer not exceeding x, and throughout we use the convention that $\sum_{j=m}^{n}=0$ if $n<m$, and hence $T_0\equiv 0$. Stigler [4] showed that, for a suitably chosen constant c, $\{T_n-nc\}/\sqrt{n}$ converges in law as $n\to\infty$ and that the limiting distribution is Gaussian if and only if a and b are continuity points of the quantile function $Q(x):=F^{-1}(x)$. The aim of the present paper is to prove a functional limit theorem for the stochastic process

$$t \rightsquigarrow X_n(t) := \frac{1}{\sqrt{n}}\{T_{[nt]}-[nt]c\}.$$

Correspondence to: Dr. Yuji Kasahara, Institute of Mathematics, University of Tsukuba, Tsukuba-shi, Ibaraki 305, Japan.

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If we write down the right-hand side using $X_j^{(n)}$, the role of the time parameter t may look less natural than in the case of Donsker's invariance principle for the usual partial sum processes of i.i.d. random variables. However, our formulation is satisfactory in order to study the limit laws of $n^{-1/2}\sup_{m\le n}|T_m - cm|$, $(1/n)\#\{m: m\le n,\ T_m \le cm\}$ and so on.

The reason why our problem cannot be treated by the usual routine work is that $X_n(\cdot)$ does not have independent increments and that, furthermore, it does not seem easy to check Chentsov-type moment conditions for tightness. So we need a new approach developed in our previous paper [2]. Our main theorem will be given in Section 3. The limiting process will be expressed as a functional of Kiefer process and is a self-similar process which does not have either independent or stationary increments in general. However, in the special case where a and b are continuity points of Q $(=F^{-1})$ and hence asymptotic normality holds by Stigler's theorem, we shall see that the limiting process is in fact a Brownian motion, which result is not surprising although it does not seem obvious, either.

Since the proof is essentially the same, we shall generalize (1.1) and be concerned with

$$T_n = T_n(f;\, a, b) := \sum_{j=[na]+1}^{[nb]} f(X_j^{(n)}), \quad n \ge 0,$$

where f is a suitable function.

2. Preliminaries

Let $S = \{S(t, u);\ t, u \in [0, 1]\}$ be a Brownian sheet and define

$$K(t, u) = S(t, u) - uS(t, 1).$$

$K = \{K(t, u);\ t, u \in [0, 1]\}$ is called a Kiefer process and may be identified as a continuous Gaussian process specified by

$$E[K(t, u)] = 0, \qquad E[K(t, u)K(t', u')] = (t \wedge t')(u \wedge u' - uu'),$$

where $a \wedge b = \min\{a, b\}$. The Kiefer process is known to be the limiting process for empirical distributions with time parameter: Let $\{\xi_j\}_{j=1}^{\infty}$ be i.i.d. random variables which are uniformly distributed over the interval $(0, 1)$ and let

$$K_n(t, u) := \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} (I_{[0,u]}(\xi_j) - u), \quad 0 \le t, u \le 1.$$

Then $K_n = \{K_n(t, u);\ 0 \le t, u \le 1\}$ $(n \ge 1)$ may be regarded as random elements of $D([0, 1]^2\colon \mathbb{R})$ endowed with S-topology (see Bickel and Wichura [1] for the definition), and it is well known that, as $n \to \infty$,

$$K_n \xrightarrow{\mathscr{L}} K \quad \text{in } D([0, 1]^2\colon \mathbb{R})$$

(see, e.g., p. 131 of Shorack and Wellner [3]). We now slightly extend this fact for our later use. Let f be a square-integrable function on the interval $(0, 1)$. Then for every fixed $t \in [0, 1]$ the Lebesgue–Stieltjes integral $W_n^f(t, s) = \int_0^s f(u)\, \mathrm{d}_u K_n(t, u)$ may be defined as usual;

$$W_n^f(t, s) = \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} \left\{ f(\xi_j) I_{[0,s]}(\xi_j) - \int_0^s f(u)\, \mathrm{d}u \right\}, \quad 0 \leq t, u \leq 1, \tag{2.1}$$

and $W_n^f = \{W_n^f(t, s);\ t, s \in [0, 1]\}$ may be regarded as a random element of $D([0, 1]^2 : \mathbb{R})$. We next consider the limiting process of W_n. For every fixed $t \in [0, 1]$, $K(t, \cdot)$ is identical in law to a Brownian bridge up to a multiplicative constant. Therefore,

$$W^f(t, \cdot) = \int_0^{\cdot} f(u)\, \mathrm{d}_u K(t, u)$$

is defined as an integration by a semimartingale for every $f \in L^2((0, 1), \mathrm{d}x)$ and in fact it is not difficult to see that $W^f = \{W^f(t, s);\ t, s \in [0, 1]\}$ admits a continuous version in two parameters. The following lemma is an easy consequence of Theorem 6 of Bickel and Wichura [1].

Lemma 2.1. *Let $f \in L^2((0, 1), \mathrm{d}x)$ and let W_n^f, W^f, K_n and K be as above. Then, as $n \to \infty$,*

$$(W_n^f, K_n) \xrightarrow{\mathcal{L}} (W^f, K) \quad \text{in } D([0, 1]^2 : \mathbb{R}^2). \qquad \square$$

Throughout the paper we shall denote $\xi_k^{([nt])}$ by $\xi_k^{(nt)}$ for typographical reason. Now by the law of large numbers we have

$$\xi_{[nts]}^{(nt)} \to s \quad \text{a.s.} \quad \text{as } n \to \infty \quad \text{for every } t, s \in (0, 1]. \tag{2.2}$$

Here notice that by monotonicity the convergence holds uniformly in s a.s. We also have a CLT for (2.2): Define

$$V_n(t, s) = \frac{[nt]}{\sqrt{n}} \{\xi_{[nts]}^{(nt)} - s\}, \quad 0 \leq t, s \leq 1. \tag{2.3}$$

Lemma 2.2. *Under the assumption of Lemma 2.1,*

$$(W_n^f, V_n, K_n) \xrightarrow{\mathcal{L}} (W^f, -K, K) \quad \text{in } D([0, 1]^2 : \mathbb{R}^3).$$

Proof. Since

$$\begin{aligned} K_n(t, \xi_{[nst]}^{(nt)}) &= \frac{1}{\sqrt{n}} \sum_{j=1}^{[nt]} \{ I(\xi_j \in [0, \xi_{[nst]}^{(nt)}]) - \xi_{[nts]}^{(nt)} \} \\ &= \frac{1}{\sqrt{n}} \{ [nst] - [nt] \xi_{[nst]}^{(nt)} \} \\ &= -V_n(t, s) + \frac{1}{\sqrt{n}} \{ [nts] - [nt]s \} \quad \text{a.s.,} \end{aligned}$$

we have

$$V_n(t, s) = -K_n(t, \xi_{[nst]}^{(nt)}) + O(1/\sqrt{n}) \quad \text{a.s.}$$

So the assertion follows from Lemma 2.1 and (2.2) by a standard argument. $\square$

Throughout the paper we denote $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$. The following result will play the key role in Section 3.

Corollary. *If $f \in L^2((0, 1), \mathrm{d}x)$, then, as $n \to \infty$,*

$$\{(W_n^f(t, \xi_{[nts]}^{(nt)}), V_n(t, s)^+, V_n(t, s)^-); t, s \in [0, 1]\}$$

$$\xrightarrow{\mathscr{L}} \{(W^f(t, s), -K(t, s)^-, -K(t, s)^+); t, s \in [0, 1]\} \quad \text{in } D([0, 1]^2: \mathbb{R}^3).$$

Proof. The assertion follows immediately from (2.2) and Lemma 2.2. $\square$

3. Main result

Let $f(x)$ be any integrable, measurable function on an interval $[0, 1]$ and we denote by $D(f)$ the totality of x such that both

$$\bar{f}(x) := \lim_{\theta \to +0} \frac{1}{\theta} \int_x^{x+\theta} f(u)\, \mathrm{d}u$$

and

$$\underline{f}(x) := \lim_{\theta \to +0} \frac{1}{\theta} \int_{x-\theta}^{x} f(u)\, \mathrm{d}u$$

exist. For convention, we define $\bar{f}(0) = f(0)$, $\underline{f}(0) = 0$, $\bar{f}(1) = 0$ and $\underline{f}(1) = f(1)$, where the right-hand sides are not at all essential but we just want to include 0 and 1 in $D(f)$ in order to simplify the statement of the theorems. Notice that if f is a function without discontinuities of the second kind, then $D(f) = [0, 1]$, $\bar{f}(x) = f(x+0)$ and $\underline{f}(x) = f(x-0)$ $(0 < x < 1)$. Now let $\{X_j^{(n)}\}_{j=1}^n$ be the order statistics of $\{X_j\}_{j=1}^n$ as in Section 1 and let $Q(s)$ denote the left-continuous inverse of the distribution function $F(x)$; i.e.,

$$Q(s) = \inf\{x: F(x) \geq s\}, \quad 0 < s \leq 1,$$
$$Q(0) = Q(0+).$$

Theorem 1. *Let $h : [0, 1] \to \mathbb{R}$ be a measurable function and let $a, b \in D(h \circ Q)$ $(0 \leq a < b \leq 1)$, and suppose $h \circ Q \in L^2((a - \epsilon, b + \epsilon) \cap [0, 1]: \mathrm{d}x)$ for some $\epsilon > 0$. Define*

$$X_n(t) = X_n^{a,b}(t) := \frac{1}{\sqrt{n}} \left\{ \sum_{j=[nat]+1}^{[nbt]} h(X_j^{(nt)}) - [nt] \int_a^b h \circ Q(x)\, \mathrm{d}x \right\} \tag{3.1}$$

and

$$X(t)=X^{a,b}(t):=\int_a^b h\circ Q(u)\,\mathrm{d}_uK(t,u)+\overline{h\circ Q}\,(b)\cdot K(t,b)^- \\ -\underline{h\circ Q}\,(b)\cdot K(t,b)^+-\overline{h\circ Q}\,(a)\cdot K(t,a)^- \\ +\underline{h\circ Q}(a)\cdot K(t,a)^+.$$

Then, as $n\to\infty$,

$$X_n \xrightarrow{\mathscr{L}} X \quad \text{in } D([0,1]:\mathbb{R}).$$

Proof. Let $\{\xi_j\}_{j=1}^{\infty}$ be as in Section 2. We shall first prove the assertion for the special case where $X_j=\xi_j$ $(j\geqslant 1)$ and hence $Q(x)\equiv x$. For the sake of simplicity we shall assume that $a=0$. The general case may be treated completely in parallel. Define

$$W_n^h(t,s)=\int_0^s h(u)\,\mathrm{d}_uK_n(t,u)$$

as in Section 2 (see (2.1)). Since $\sum_{j=1}^{[nbt]} h(\xi_j^{(nt)})=\sum_{j=1}^{[nt]} h(\xi_j)I(\xi_j\leqslant \xi_{[nbt]}^{(nt)})$ a.s., we have the following equation which plays the key role in this paper.

$$\begin{aligned}
X_n(t)&=\frac{1}{\sqrt{n}}\left\{\sum_{j=1}^{[nbt]} h(\xi_j^{(nt)})-[nt]\int_0^b h(u)\,\mathrm{d}u\right\}\\
&=\frac{1}{\sqrt{n}}\left\{\sum_{j=1}^{[nt]} h(\xi_j)I(\xi_j\leqslant \xi_{[nbt]}^{(nt)})-[nt]\int_0^{\xi_{[nbt]}^{(nt)}} h(u)\,\mathrm{d}u\right.\\
&\qquad\left.+[nt]\int_b^{\xi_{[nbt]}^{(nt)}} h(u)\,\mathrm{d}u\right\}\\
&=W_n^h(t,\xi_{[ntb]}^{(nt)})+\frac{[nt]}{\sqrt{n}}\int_b^{\xi_{[nbt]}^{(nt)}} h(u)\,\mathrm{d}u\\
&=W_n^h(t,\xi_{[ntb]}^{(nt)})+\frac{[nt]}{\sqrt{n}}\bar{h}(b)(\xi_{[ntb]}^{(nt)}-b)^+-\frac{[nt]}{\sqrt{n}}\underline{h}(b)(\xi_{[ntb]}^{(nt)}-b)^-+\epsilon_n(t)\\
&=W_n^h(t,\xi_{[ntb]}^{(nt)})+\bar{h}(b)V_n(t,b)^+-\underline{h}(b)V_n(t,b)^-+\epsilon_n(t),
\end{aligned}\tag{3.2}$$

where

$$\epsilon_n(t)=\frac{[nt]}{\sqrt{n}}\left\{\int_b^{\xi_{[ntb]}^{(nt)}} h(u)\,\mathrm{d}u-\bar{h}(b)(\xi_{[ntb]}^{(nt)}-b)^++\underline{h}(b)(\xi_{[ntb]}^{(nt)}-b)^-\right\}.\tag{3.3}$$

(See (2.3) for the definition of V_n.) We shall first see that ϵ_n is negligible: If $b=1$ then the assertion of the theorem is obvious. If $b<1$ then by the definition of $\bar{h}(b)$ and $\underline{h}(b)$, we see that, for every $\varepsilon>0$, there exists a $\delta>0$ such that

$$\left|\int_b^{b+\theta} h(x)\,\mathrm{d}x-\bar{h}(b)\theta^++\underline{h}(b)\theta^-\right|\leqslant\varepsilon|\theta|\quad \text{if } |\theta|\leqslant\delta.\tag{3.4}$$

Let

$$\Omega_m^\delta = \{\omega : |\xi_{[kb]}^{(k)}(\omega) - b| \leqslant \delta \ \forall k \geqslant m\}, \quad m \geqslant 1. \tag{3.5}$$

Notice that, by the law of large numbers (see (2.2)), we have

$$\lim_{m\to\infty} P(\Omega_m^\delta) = 1, \quad \delta > 0. \tag{3.6}$$

If $\omega \in \Omega_m^\delta$, then by definition we have

$$|\xi_{[ntb]}^{(nt)} - b| \leqslant \delta \quad \text{for all } t \geqslant m/n \ (n \geqslant 1).$$

Therefore, it follows from (3.3) and (3.4) that

$$|\epsilon_n(t)| \leqslant \frac{[nt]}{\sqrt{n}}\, \varepsilon \cdot |\xi_{[ntb]}^{(nt)} - b| \leqslant \varepsilon |V_n(t, b)|, \quad t \geqslant m/n, \ n \geqslant 1, \ \omega \in \Omega_m^\delta,$$

and hence

$$\sup_{m/n \leqslant t \leqslant 1} |\varepsilon_n(t)| \leqslant \varepsilon \sup_{0 \leqslant t \leqslant 1} |V_n(t, b)|, \quad \omega \in \Omega_m^\delta.$$

Consequently, we have

$$P\left(\sup_{0\leqslant t\leqslant 1} |\epsilon_n(t)| > \eta\right)$$
$$\leqslant P\left(\sup_{0 \leqslant t \leqslant m/n} |\epsilon_n(t)| > \eta\right) + P\left(\varepsilon \sup_{0\leqslant t\leqslant 1} |V_n(t, b)| > \eta\right) + P(\omega \notin \Omega_m^\delta).$$

Now letting $n \to \infty$ we have from Lemma 2.2 that

$$\limsup_{n\to\infty} P\left(\sup_{0\leqslant t\leqslant 1} |\epsilon_n(t)| \geqslant \eta\right) \leqslant 0 + P\left(\varepsilon \sup_{0\leqslant t\leqslant 1} |K(t, b)| \geqslant \eta\right) + P(\omega \notin \Omega_m^\delta).$$

(The first term of the right-hand side is obtained easily from (3.3).) Since m is arbitrary, letting $m \to \infty$, we have from (3.6) that

$$\limsup_{n\to\infty} P\left(\sup_{0\leqslant t\leqslant 1} |\epsilon_n(t)| \geqslant \eta\right) \leqslant P\left(\varepsilon \sup_{0\leqslant t\leqslant 1} |K(t, b)| \geqslant \eta\right).$$

Letting $\varepsilon \downarrow 0$, we conclude

$$\sup_{0\leqslant t\leqslant 1} |\epsilon_n(t)| \xrightarrow{\mathrm{P}} 0 \quad \text{as } n \to \infty.$$

Let us go back to (3.2). As we have seen above, the error term ϵ_n in (3.2) is negligible. We see from the Corollary of Lemma 2.2 that the joint distribution of the first three terms in the extreme right-hand side of (3.2) converges in law to

$$(W^h(\cdot, b), -\bar{h}(b)K(\cdot, b)^-, \underline{h}(b)K(\cdot, b)^+)$$

in $D([0, 1]: \mathbb{R}^3)$. Therefore, we conclude that

$$X_n \xrightarrow{\mathscr{L}} X = \{W^h(t, b) - \bar{h}(b)K(t, b)^- + \underline{h}(b)K(t, b)^+;\ t \in [0, 1]\}.$$

Thus the assertion of the theorem is proved for the special case where $X_j = \xi_j$ ($j \geq 1$) and where h is square-integrable. The general case may easily be reduced to this case by a standard argument as follows: Since $Q(\xi_j)$ ($Q = F^{-1}$) is identical in law to X_j, we see that $\{h(X_j)\}_{j=1}^{\infty}$ is distributed like $\{h \circ Q(\xi_j)\}_{j=1}^{\infty}$. Therefore, we have the assertion replacing h by $h \circ Q$ provided that $h \circ Q \in L^2([0, 1], dx)$, which restriction may easily be removed using the usual cut-off method (i.e., consider $h \circ QI_{(a-\epsilon, b+\epsilon)}$, which is square-integrable by assumption, in place of $h \circ Q$ itself. For details see [2]). □

Theorem 2. *If, in addition to the assumptions of Theorem* 1, *a and b are continuity points of $h \circ Q$, then*

$$X_n \xrightarrow{\mathscr{L}} \sigma B \quad \text{in } D([0, 1]: \mathbb{R})$$

where $B = (B(t))_{t \geq 0}$ is a standard Brownian motion and $\sigma^2 = E[X^{a,b}(1)^2]$.

Proof. By the assumption that a and b are continuity points of $h \circ Q$, we see from Theorem 1 that the limiting process is

$$X(t) = \int_a^b h \circ Q(u) \, d_u K(t, u) - h \circ Q(b) \cdot K(t, b) + h \circ Q(a) \cdot K(t, a).$$

Since $t \rightsquigarrow K(t, \cdot)$ may be regarded as a $C([0, 1]: \mathbb{R})$-valued process with independent increments, we easily see that $X = \{X(t);\, t \in [0, 1]\}$ is a centered Gaussian process with stationary independent increments. Therefore, we obtain the assertion. □

Remark. If $h \circ Q$ is continuous and of bounded variation over $[a, b]$, then $X(t)$ in Theorem 2 has a simpler expression as follows after an integration by parts.

$$X(t) = -\int_a^b K(t, u) \, d(h \circ Q)(u),$$

and hence

$$\sigma^2 = \int_a^b \int_a^b (u \wedge v - uv) \, d(h \circ Q)(u) \, d(h \circ Q)(v).$$

So far we considered $X_n(t)$ ($= X_n^{a,b}(t)$) for fixed a and b. But as in [2] we may also consider that it is a process with multi-parameter (b, t) for a fixed a. Keeping in mind that (3.4) holds uniformly for b over an interval I on which h is continuous, we see that the proof of Theorem 1 is still valid uniformly for $b \in I$. Therefore, we have:

Theorem 3. *If, in addition to the assumption of Theorem* 1, *$h \circ Q$ is continuous on $(a - \epsilon, b + \epsilon) \cap [0, 1]$, then*

$$Z_n = \{X_n^{a,u}(t);\, 0 \leq t, u \leq 1\} \xrightarrow{\mathscr{L}} Z = \{X^{a,u}(t);\, 0 \leq t, u \leq 1\} \quad \text{in } D([0, 1] \times [a, b]: \mathbb{R}). \qquad \square$$

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